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EXTERNAL CRACK IN TORSION IN AN INFINITE MEDIUM WITH A CYLINDRICAL INCLUSION†

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Abstract—This paper deals with the stress distribution in an infinite homogeneous isotropic elastic medium with a cylindrical inclusion and an external crack opened by internal shear stress acting along the crack. The cylindrical inclusion is also assumed to be homogeneous isotropic and elastic. The problem is reduced to the solution of a Fredholm integral equation which is solved numerically. Closed form expression is obtained for the stress intensity factor and its numerical values are graphed to demonstrate the effect of the modulus of rigidity of the inclusion. The order of the stress singularity is obtained when crack touches the cylindrical inclusion.

1. INTRODUCTION

Mixed boundary value problems for an infinite medium with a cylindrical cavity have been considered by Srivastav[1, 2], Srivastav and Lee[3] and Srivastav and Narain[4]. The solutions to the problems of an external crack in an infinite medium under torsion or an infinite medium with a cylindrical hole whose surface is rigidily fixed are given by Kassir and Sih[5]. The problem of a penny-shaped crack in a cylindrical inclusion in an infinite medium has been considered by Freeman and Keer[6]. Arutiunian and Babloian[7] have considered the problems of torsion by a circular die of an infinite medium with a circular hole and an infinite medium with a circular inclusion, when the radius of the die is greater than the radius of the circular hole or inclusion.

We consider the problem of an infinite medium (matrix) of modulus of rigidity G_2 with an infinite cylindrical inclusion of modulus of rigidity G_1 when the infinite medium is twisted by equal and opposite couples. This loading is assumed to be simulated by the application of a variable circumferential shearing force $p(r)$ to the external crack surfaces. A perfect bond is assumed between the two materials. By assuming appropriate solutions for the matrix and the inclusion, we reduce the problem to the solution of dual integral equations. The dual integral equations are further reduced to the solution of a Fredholm integral equation for the determination of an auxiliary function. The stress intensity factor is obtained in terms of the auxiliary function. The numerical values of the integral equation and the stress intensity factor are obtained for $p(r) = -1/r^2$. These numerical values of the stress intensity factor are graphed to demonstrate the effect of the modulus of rigidity of the inclusion. The analytical solution obtained is shown to be in complete agreement with the corresponding solutions obtained by Kassir and Sih[5] for the problem of an external crack in an infinite medium ($G = 1$) and in an infinite medium with a rigidly fixed cylindrical hole ($G = 0$). Finally order of the singularity is obtained when the crack approaches the inclusion (i.e. $a = b$). It may be realistic to say that the analysis will be applicable only for $p(r) \sim r^{-\beta}$ ($\beta \geq 1$).

2. BASIC EQUATIONS AND THEIR SOLUTION

We consider the infinite space consisting of two materials, under torsion. The two materials are bonded along the surface $r = 1$ and both the materials are assumed to be homogeneous isotropic and elastic. In the case of axial symmetry the displacement vector $\vec{u}$ assumes the form $(0, v, 0)$ in the cylindrical coordinate system (r, θ, z). The equation of equilibrium in this case is given by:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{3}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = 0, \tag{1}$$

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where the displacement component v and the non-zero stress components $\sigma_{r\theta}$ and $\sigma_{z\theta}$ are given in terms of the displacement function $\Phi(r, z)$ by the formulae:

$$v = r\Phi(r, z), \qquad \sigma_{r\theta} = Gr\frac{\partial\Phi}{\partial r}, \qquad \sigma_{z\theta} = Gr\frac{\partial\Phi}{\partial z}, \tag{2}$$

G being the shear modulus of the material.

For regions 1 $(0 \le r < b,\ -\infty < z < \infty)$ and 2 $(r > b,\ -\infty < z < \infty)$ we take the following Fourier–Bessel integral representations for the function Φ (see Sneddon[8]):

$$\Phi^{(1)}(r, z) = \frac{1}{r}\int_0^\infty A(\xi)I_1(\xi r)\sin\xi z\,\mathrm{d}\xi, \tag{3}$$

$$\Phi^{(2)}(r, z) = \frac{1}{r}\int_0^\infty [B(\xi)J_1(\xi r)\,\mathrm{e}^{-\xi z} + C(\xi)K_1(\xi r)\sin\xi z]\,\mathrm{d}\xi, \tag{4}$$

where A, B and C are arbitrary functions of ξ.

Substituting for $\Phi^{(1)}$ and $\Phi^{(2)}$ respectively from (3) and (4) into eqns (2) we obtain the following expressions the displacement and stress components for the regions 1 and 2 respectively:

$$v^{(1)}(r, z) = \int_0^\infty A(\xi)I_1(\xi r)\sin\xi z\,\mathrm{d}\xi, \tag{5}$$

$$\sigma_{z\theta}^{(1)}(r, z) = G_1\int_0^\infty \xi A(\xi)I_1(\xi r)\cos\xi z\,\mathrm{d}\xi, \tag{6}$$

$$\sigma_{r\theta}^{(1)}(r, z) = G_1\int_0^\infty \xi A(\xi)I_2(\xi r)\sin\xi z\,\mathrm{d}\xi, \tag{7}$$

and

$$v^{(2)}(r, z) = \int_0^\infty [B(\xi)J_1(\xi r)\,\mathrm{e}^{-\xi z} + C(\xi)K_1(\xi r)\sin\xi z]\,\mathrm{d}\xi, \tag{8}$$

$$\sigma_{z\theta}^{(2)}(r, z) = G_2\int_0^\infty \xi[-B(\xi)J_1(\xi r)\,\mathrm{e}^{-\xi z} + C(\xi)K_1(\xi r)\cos\xi z]\,\mathrm{d}\xi, \tag{9}$$

$$\sigma_{r\theta}^{(2)}(r, z) = -G_2\int_0^\infty \xi[B(\xi)J_2(\xi r)\,\mathrm{e}^{-\xi z} + C(\xi)K_2(\xi r)\sin\xi z]\,\mathrm{d}\xi, \tag{10}$$

where G_1 and G_2 are the shear moduli for the regions 1 and 2 respectively, $J_n(x)$ is a Bessel function of the first kind and $I_n(x)$ and $K_n(x)$ are modified Bessel functions of the first and second kinds respectively.

3. STATEMENT OF THE PROBLEM AND DERIVATION OF THE INTEGRAL EQUATION

If we have an axisymmetric crack in the infinite medium located at $z = 0$, $r > a > b$ which is opened by a shear stress acting on the crack surfaces, the boundary conditions for the problem may be taken as:

$$\sigma_{z\theta}^{(2)}(r, 0) = p(r), \quad a < r < \infty, \tag{11}$$

$$v^{(2)}(r, 0) = 0, \quad b < r < a, \tag{12}$$

$$v^{(1)}(r, 0) = 0, \quad 0 < r < b. \tag{13}$$

In addition to the above boundary conditions we have the following continuity conditions along the surface $r = b$:

$$\sigma_{r\theta}^{(1)}(b, z) = \sigma_{r\theta}^{(2)}(b, z), \tag{14}$$

$$v^{(1)}(b, z) = v^{(2)}(b, z). \tag{15}$$

From (5), we see that the boundary condition (13) is identically satisfied. Using (8) and (9), we find that the boundary conditions (11) and (12) lead to the following dual integral equations:

$$\int_0^\infty B(\xi)J_1(\xi r)\,d\xi = 0, \quad b < r < a, \tag{16}$$

$$\int_0^\infty \xi B(\xi)J_1(\xi r)\,d\xi = g(r), \quad r > a, \tag{17}$$

where

$$g(r) = -\frac{p(r)}{G_2} + \int_0^\infty \xi C(\xi)K_1(\xi r)\,d\xi. \tag{18}$$

If we introduce the representation

$$B(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \int_a^\infty t\psi(t)\sin \xi t\,dt, \quad \psi(\infty) = 0, \tag{19}$$

we find that the integral eqn (16) is identically satisfied and the integral eqn (17) leads to

$$\frac{\partial}{\partial r}\int_r^\infty \frac{t\psi(t)\,dt}{(t^2 - r^2)^{1/2}} = -\left(\frac{\pi}{2}\right)^{1/2} g(r), \quad r > a. \tag{20}$$

The solution of the above integral equation is given by

$$\psi(t) = \left(\frac{2}{\pi}\right)^{1/2} \int_t^\infty \frac{g(r)}{(r^2 - t^2)^{1/2}}\,dr, \quad t > a. \tag{21}$$

Substituting the value of $g(r)$ from (18) into (21) and making use of the result

$$\int_t^\infty \frac{K_1(\xi r)}{(r^2 - t^2)^{1/2}}\,dr = \frac{\pi}{2\xi t}\,e^{-\xi t}, \tag{22}$$

we obtain

$$\psi(t) = \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{t}\int_0^\infty C(\xi)\,e^{-\xi t}\,d\xi - \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{G_2}\int_t^\infty \frac{p(r)\,dr}{(r^2 - t^2)^{1/2}}, \quad t > a. \tag{23}$$

From eqns (5), (7), (8), (10), (14) and (15), we obtain the following two equations for the determination of $A(\xi)$ and $C(\xi)$ in terms of $B(\xi)$:

$$\int_0^\infty [GC(\xi)K_2(b\xi) + A(\xi)I_2(b\xi)]\xi \sin \xi z\,d\xi = -G\int_0^\infty \xi B(\xi)J_2(b\xi)\,e^{-\xi z}\,d\xi, \tag{24}$$

$$\int_0^\infty [C(\xi)K_1(b\xi) - A(\xi)I_1(b\xi)] \sin \xi z\,d\xi = -\int_0^\infty B(\xi)J_1(b\xi)\,e^{-\xi z}\,d\xi, \tag{25}$$

where $G = G_2/G_1$.

Inverting the fourier sine transform in eqns (24) and (25) and using the following result

$$\int_0^\infty e^{-\xi z} \sin \zeta z \,dz = \frac{\zeta}{\xi^2 + \zeta^2}, \tag{26}$$

we obtain

$$GC(\xi)K_2(b\xi) + A(\xi)I_2(b\xi) = L_1(\xi), \tag{27}$$

$$C(\xi)K_1(b\xi) - A(\xi)I_1(b\xi) = L_2(\xi), \tag{28}$$

where

$$L_1(\xi) = -\frac{2G}{\pi}\int_0^\infty \frac{\zeta B(\zeta)J_2(b\zeta)\,d\zeta}{\xi^2+\zeta^2}, \quad L_2(\xi) = \frac{-2\xi}{\pi}\int_0^\infty \frac{B(\zeta)J_1(b\zeta)\,d\zeta}{\xi^2+\zeta^2}. \tag{29}$$

Substituting for $B(\zeta)$ from (19) into (29), we find that

$$L_1(\xi) = \left(\frac{2}{\pi}\right)^{1/2} GI_2(b\xi)\int_a^\infty t\psi(t)\,e^{-\xi t}\,dt, \tag{30}$$

$$L_2(\xi) = -\left(\frac{2}{\pi}\right)^{1/2} I_1(b\xi)\int_a^\infty t\psi(t)\,e^{-\xi t}\,dt. \tag{31}$$

From (27), (28), (30) and (31) we obtain

$$C(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(G-1)I_1(b\xi)I_2(b\xi)}{K_1(b\xi)I_2(b\xi) + GK_2(b\xi)I_1(b\xi)}\int_a^\infty x\psi(x)\,e^{-\xi x}\,dx. \tag{32}$$

Substituting for $C(\xi)$ from (32) into (23), we obtain the following Fredholm integral equation of the second kind for the determination of $\psi(t)$:

$$\psi(t) + \int_a^\infty K(x,t)\psi(x)\,dx = f(t), \quad t > a, \tag{33}$$

where

$$K(x,t) = \frac{1-G}{t}\int_0^\infty \frac{xI_1(b\xi)I_2(b\xi)\,e^{-(x+t)\xi}\,d\xi}{K_1(b\xi)I_2(b\xi) + GK_2(b\xi)I_1(b\xi)}, \tag{34}$$

$$f(t) = -\frac{1}{G_2}\left(\frac{2}{\pi}\right)^{1/2}\int_t^\infty \frac{p(r)\,dr}{(r^2-t^2)^{1/2}}. \tag{35}$$

If we take $G_1 = G_2$ (i.e. $G = 1$), we find that $K(x,t) \equiv 0$ and that

$$\psi(t) = f(t) = -\frac{1}{G_2}\left(\frac{2}{\pi}\right)^{1/2}\int_t^\infty \frac{p(r)\,dr}{(r^2-t^2)^{1/2}}, \tag{36}$$

which is the solution for the problem of an external crack in an infinite medium under torsion.

Substituting for $\psi(t)$ from (36) into (19), we obtain

$$B(\xi) = -\frac{2}{\pi G_2}\int_a^\infty t \sin \xi t\,dt \int_t^\infty \frac{p(r)\,dr}{(r^2-t^2)^{1/2}}, \tag{37}$$

which is in complete agreement with eqn (2.20) of Kassir and Sih[5]. If we take $G = 0$ (i.e. $G_1 \to \infty$), we find that $\psi(t)$ is given by the integral eqn (33) where

$$K(x,t) = \frac{x}{t}\int_0^\infty \frac{I_1(b\xi)}{K_1(b\xi)}\,e^{-(x+t)\xi}\,d\xi, \tag{38}$$

which is in complete agreement with the known solution of the corresponding problem of an external crack opened by torsional loading in an infinite medium with a cylindrical cavity whose surface is rigidly fixed (see Kassir and Sih[5], pp. 328–329).

4. STRESS INTENSITY FACTOR AND NUMERICAL RESULTS

From eqn (9) we find that

$$\sigma_{z\theta}^{(2)}(r,0) = -G_2\left[\int_0^\infty \xi B(\xi) J_1(\xi r)\,\mathrm{d}\xi - \int_0^\infty \xi C(\xi) K_1(\xi r)\,\mathrm{d}\xi\right]. \tag{39}$$

Integrating (19) by parts, we obtain

$$B(\xi) = \frac{1}{\xi}\left(\frac{2}{\pi}\right)^{1/2}\left[a\psi(a)\cos a\xi + \int_a^\xi \{t\psi(t)\}'\cos \xi t\,\mathrm{d}t\right] \tag{40}$$

where the prime denotes the differentiation with respect to t. Substituting for $B(\xi)$ and $C(\xi)$ respectively from (40) and (32) into (39), we obtain

$$\sigma_{z\theta}^{(2)}(r,0) = -\left(\frac{2}{\pi}\right)^{1/2} G_2\left[\frac{1}{r}\left(1-\frac{a}{\sqrt{(a^2-r^2)}}\right)a\psi(a) + \frac{1}{r}\int_a^\infty \{t\psi(t)\}'\left(1-\frac{t}{\sqrt{(t^2-r^2)}}\right)\mathrm{d}t\right.$$
$$\left. -(G-1)\int_a^\infty t\psi(t)\,\mathrm{d}t\int_0^\infty \frac{\xi K_1(\xi r) I_1(b\xi) I_2(b\xi)\,e^{-t\xi}\,\mathrm{d}\xi}{K_1(b\xi)I_2(b\xi)+GK_2(b\xi)I_1(b\xi)}\right],\quad b<r<a. \tag{41}$$

The stress intensity factor K at the edge of the crack $r=a$ is given by

$$K = \lim_{r\to a-}\left[\{2(a-r)\}^{1/2}\sigma_{\theta z}^{(2)}(r,0)\right]. \tag{42}$$

From (41) and (42), we find that

$$K = G_2\left(\frac{2a}{\pi}\right)^{1/2}\psi(a). \tag{43}$$

For the case of infinite medium without cylindrical inclusion $\psi(t)$ is given by eqn (36) and hence for this case

$$K = -\frac{2}{\pi}a^{1/2}\int_a^\infty \frac{p(r)\,\mathrm{d}r}{(r^2-a^2)^{1/2}}, \tag{44}$$

which agrees with eqn (2.23) of Kassir and Sih ([5], p. 49). For the case

$$p(r) = -\frac{1}{r^2}, \tag{45}$$

we find that

$$f(t) = \frac{1}{G_2}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{t^2}, \tag{46}$$

and hence the integral eqn (33) may be written in the following form:

$$\phi(t) + \int_a^\infty K(x,t)\phi(x)\,\mathrm{d}x = 1/t^2,\quad a<t<\infty, \tag{47}$$

where

$$\phi(t) = G_2\left(\frac{\pi}{2}\right)^{1/2}\psi(t). \tag{48}$$

From (48) and (43), we have

$$K = \frac{2}{\pi}a^{1/2}\phi(a). \tag{49}$$

Integral eqn (47) is solved numerically for $b = 1$ and $a = 1.2\,(0.1)\,2.0$ for $G = 0, 0.05, 0.2, 1.0, 50, 100, 1000$. The numerical values of K are then obtained from eqn (49) and these are graphed in Fig. 1. It is noticed that the values of the stress intensity factor decrease with the increase of the value of the modulus of rigidity of the cylindrical inclusion. In particular the minimum value of the stress intensity factor occurs when the inclusion is rigid ($G = 0$). For the case when the modulus of rigidity of the inclusion is smaller than the modulus of rigidity of the surrounding material ($G = G_2/G_1 > 1$) there is very little effect on the stress intensity factor. The values of K are obtained for upto $G = 10^3$ but they differ from the values of K for $G = 50$ only in the fourth decimal place.

5. STRESS SINGULARITY

It is noted that when $a = b = 1$, the kernel of the Fredholm integral (33) becomes unbounded. The stress singularity for this case can be checked by the technique used by Keer and Freeman[9] and Freeman and Keer[6] who have discussed the related problems. Using their method we shall find the stress singularity for $a = b = 1$, from the integral eqn (33) directly. If $\psi(t)$ is assumed in the form

$$\psi(t) \approx (t-1)^{-k} \quad \text{as} \quad t \to 1^+,$$

where k is a constant the stress may be shown to have the form

$$\sigma^2_{z\theta}(r, 0) \approx (1-\rho)^{-k-1/2} \quad \text{as} \quad \rho \to 1^-.$$

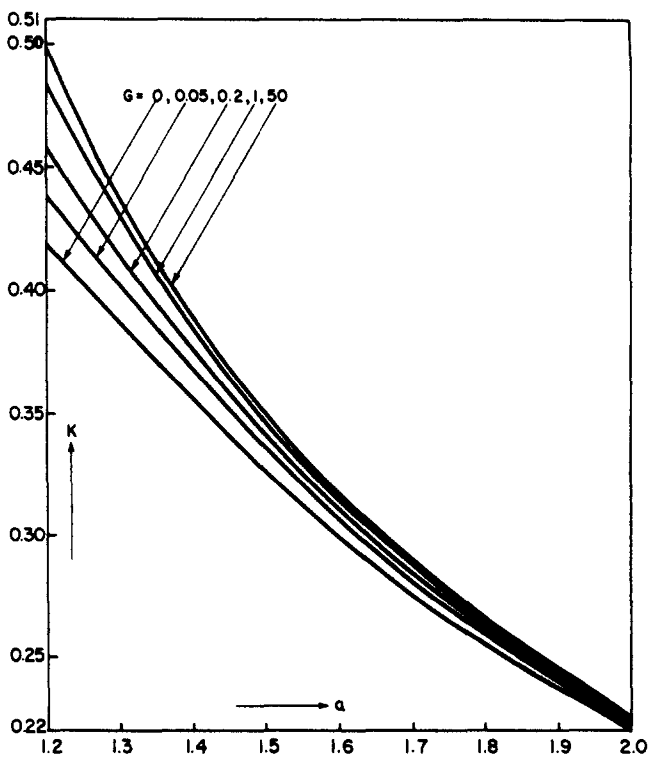

Fig. 1.

To illustrate the singular behavior of the integral eqn (33) we consider the limiting case of the kernel $K(x, t)$ when $\xi \to \infty$. Thus, if in eqn (33) the asymptotic form of the kernel as $\xi \to \infty$ is added and subtracted we find that

$$t\psi(t) + \frac{1}{\pi}\left(\frac{1-G}{1+G}\right)\int_1^\infty \frac{x\psi(x)\,dx}{(x+t-2)} + H(t) = 0, \tag{50}$$

where $H(t)$ is non-singular as $t \to 1$.

If a test function of the type

$$\psi(t) = \frac{A}{t}(t-1)^{-k}, \tag{51}$$

is substituted into eqn (50) then we find that k is governed by the equation

$$\sin(\pi k) = \frac{1-G}{G+1}. \tag{52}$$

Equation (52) implies that the stress intensity factor is zero, except for the special case $G = 1$, which corresponds to $k = 0$. Now it is also clear that the stress singularity depends upon G and that the order of the singularity is between zero and one.

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